Vermont State Mathematics Coalition Talent Search -- January 2018
Test 3 of the 2017-2018 school year

PRINT NAME:____ Signature:_________________
Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

School____ Grade _____________

Current Mathematics Teacher: ____

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@fnwsu.org or be postmarked by **February 16, 2018** and submitted to

Kiran MacCormick
Missisquoi Valley Union High School
175 Thunderbird Drive
Swanton, VT 05488

To receive the next tests via email, clearly print your email address below:

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The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Missisquoi Valley Union HS) and Evan Dummit (Postdoctoral Associate at Arizona State University).

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1. Nick has 2 identical cups: the first cup is full of water, while the second is empty. He pours half the water from the first into the second. Then, on transfer 2, he pours ⅓ of the water in the 2nd cup back into the first cup. He repeats this, alternating cups, pouring $1/(i+1)$ of the water in a cup back into the other cup on the ith transfer. What fraction of the water is in the first cup just after the 2018th transfer?

Answer:___________

2. Suppose you are a sportsbook taking 100 bets for who will win the Super Bowl. If each bettor picks exactly one team from the Vikings, Jaguars, Eagles and Patriots, how many different combinations of wagers are possible for the 100 bets you are taking? Here are some examples of possible combinations. You do not distinguish between individual bettors.

(i) V = 30, J = 15, E = 40, P = 15.
(ii) V = 8, J = 32, E = 60, P = 0.
(iii) V = 0, J = 0, E = 0, P = 100.

Answer:___________

3. What is the length of the shortest path AQRB in the plane, where A = (14, 12), B = (29, 1), Q lies on the y-axis and R lies on the x-axis?

Answer:___________

4. If $0 < \theta < \frac{\pi}{2}$ and $\frac{1 + 2\sin(\theta) + 3\sin^2(\theta) + 4\sin^3(\theta) + \ldots}{1 + 2\cos(\theta) + 3\cos^2(\theta) + 4\cos^3(\theta) + \ldots} = \frac{4}{81}$,

find the value of $1 + 2\tan(\theta) + 3\tan^2(\theta) + 4\tan^3(\theta) + \ldots$.

Answer:___________

5. Triangle ABC has AB = 22, AC = 24, and BC = 26, and line l bisects both the perimeter and area of ABC. If line l intersects triangle ABC in points D and E, find all possible lengths of segment DE.

Answer:______________________________

6. Let $p(x)$ be a monic polynomial with integer coefficients. We call a triangle p-special if its 3 vertices all have integer coordinates and lie on the graph of $y=p(x)$, and we say the positive integer n is p-special if there is a p-special triangle whose area is n.
a) Show that there exists a polynomial p of degree 3 such that the p-special integers are precisely the positive multiples of 3.

b) Determine, with proof, whether there exists a polynomial p of degree 3 such that every positive integer is p-special.

Note: For this problem, please include your proof on separate sheets of paper.

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