Vermont Mathematics Talent Search, Solutions to Test 2, 2017-2018

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1. Two circles with centers A and B have radii of 8 inches and 20 inches, respectively. The centers are 35 inches apart. If CD is a common internal tangent to the two circles and CD intersects AB at E, what is the length of segment AE?

   Answer: 10.

   Solution: Upon drawing radii AC and BD, we see that right triangles ACE and BDE are similar, since the angles AEC and BED are equal. Since $AC = 8$ and $BD = 20$, we conclude that $\frac{AE}{BE} = \frac{8}{20}$, so that $BE = \frac{5}{2}AE$. Then because $AE + BE = 35$, we obtain $AE + \frac{5}{2}AE = 35$ so that $\frac{7}{2}AE = 35$. Therefore, $AE = \boxed{10}$.

2. Find the number of perfect cubes in the set $\{1^1, 2^2, 3^3, \ldots, 2017^{2017}\}$.

   Answer: 680.

   Solution: We observe that k^k will be a perfect cube either when k is divisible by 3 or when k is itself a perfect cube. For any k not of this form, there is some prime p in the prime factorization of k whose exponent is not divisible by 3: then the exponent of p in the prime factorization of k^k will be k times as large, and thus still not divisible by 3.

   There are $\lfloor 2017/3 \rfloor = 672$ values of k that are multiples of 3, and $\lfloor \sqrt[3]{2017} \rfloor = 12$ values of k that are perfect cubes, but 4 of these (namely, $k = 3^3$, 6^3, 9^3, and 12^3) are in both categories. Thus, the number of perfect cubes is $672 + 12 - 4 = \boxed{680}$.

3. Find all real solutions to the equation $2\log(x + 16) = 2 + \log(x)$.

   Answer: $x = 4$, 64.

   Solution: Suppose that $2\log(x+16) = 2+\log(x)$. By applying properties of logarithms, this equation implies that $\log\left[(x + 16)^2\right] = \log(100x)$, from which we conclude that $(x+16)^2 = 100x$, so that $x^2-68x+256 = 0$. Factoring yields $(x - 4)(x - 64) = 0$, so that $x = \boxed{4,\ 64}$.

   Remark: Note that both values do in fact satisfy the original equation: $2\log(20) = \log(400) = 2+\log(4)$, and $2\log(80) = \log(6400) = 2+\log(64)$. (In general, this step is necessary since the statement $2\log(x+16) = \log(x + 16)^2$ is only true when $x + 16 > 0$, so it would have produced extraneous solutions had $x + 16$ been negative for any of the resulting values of x.)

4. A 2017-dimensional hypercube of side length 1 is drawn along with all of its diagonals, yielding a total of $2^{4033} - 2^{2016}$ segments. Compute the median length of these segments.

   Answer: $\sqrt{1009}$.

   Solution: Embed the cube in space so that its vertices have all coordinates either 0 or 1. By symmetry, the number of segments of a given length in the entire cube will be 2^{2016} times the number of segments of a given length starting at a particular vertex A of the cube, and so we may equivalently find the median length of a segment starting at the vertex $(0, 0, \cdots, 0)$. A diagonal with one endpoint at $(0, 0, \ldots, 0)$ will have length $\sqrt{k}$ precisely when the coordinates of the other endpoint consist of exactly k ones and $2017 - k$ zeroes, and therefore there are $\binom{2017}{k}$ segments of length $\sqrt{k}$.

We therefore must compute the median of the list containing $\binom{2017}{k}$ copies of $\sqrt{k}$, for $1 \le k \le 2017$. By symmetry, since $\sqrt{k}$ and $\sqrt{2017-k}$ each appear the same number of times for $1 \le k \le 2016$, and since $\sqrt{2017}$ appears on the list but 0 does not, the median is $\sqrt{(2017+1)/2} = \boxed{\sqrt{1009}}$.

5. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions: (i) $f(f(n)) = n$ for all integers n; (ii) $f(f(n+2)+2) = n$ for all integers n; (iii) $f(0) = 1$.

Remark: Many solvers assumed that $f(n)$ would necessarily be a polynomial in n. However, this cannot be assumed from the fact that f is a function from the integers to the integers: for example, $f(n) = |n|$ is also a function from the integers to the integers, but is not a polynomial. Another such function is $g(n) = \begin{cases} \sqrt{n} & \text{if } n \text{ is a perfect square} \\ 2n+5 & \text{otherwise} \end{cases}$.

Solution 1: By applying (i) with $n = 0$, we see that $0 = f(f(0)) = f(1)$ so that $f(1) = 0$. Furthermore, by applying (i) and (ii) appropriately, we can write

$$f(n) \overset{\text{(ii)}}{=} f\left[f(f(n+2)+2)\right] = f(f[f(n+2)+2]) \overset{\text{(i)}}{=} f(n+2)+2$$

from which we deduce $f(n) = f(n+2)+2$ for every integer n, so that $f(n+2) = f(n) - 2$.

Now we use induction to show that $f(n) = 1 - n$ for all integers n. We take $n = 0, 1$ as our base cases: as calculated above, we have $f(1) = 0$ and $f(0) = 1$. For the inductive step, suppose $f(2k) = 1 - 2k$ and $f(2k+1) = 1-(2k+1)$: then $f(2k+2) = f(2k)-2 = 1-(2k+2)$ and $f(2k+3) = f(2k+1)-2 = 1-(2k+3)$, as required. In the same way, we see that $f(2k-2) = 1-(2k-2)$ and $f(2k-1) = 1-(2k-1)$.

Thus, by induction, $f(n) = 1 - n$ for all integers n.

Solution 2: If $f(a) = f(b)$ for some integers a and b, then by (i), we see $a = f(f(a)) = f(f(b)) = b$ so that f is one-to-one. Then since $f(f(n)) = n = f(f(n+2)+2)$, we must have $f(n) = f(n+2)+2$ for every integer n. The rest then follows as in Solution 1.

6. A set of 15 (distinct) positive integers has the property that the arithmetic mean of the elements in any nonempty subset is an integer. Find the smallest possible value for the largest of the integers.

Answer: 5045041.

Solution: Since any collection of k of the integers has an integral arithmetic mean, their sum must be divisible by k.

If a and b are any two of the integers, then for any $1 \le k \le 14$, select any set S of $k-1$ of the other integers: then the sets $S \cup \{a\}$ and $S \cup \{b\}$ both have sum divisible by k, meaning that k divides $b - a$.

Therefore, if the integers are $a_1 < a_2 < \cdots < a_{15}$, then $a_{i+1} - a_i$ is divisible by each of 2, 3, ... , 14 and hence by their least common multiple, which is $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 360360$.

Then $a_1 \ge 1$, $a_2 \ge 1 + 360360$, $a_3 \ge 1 + 2 \cdot 360360$, and in general, $a_i \ge 1 + (i-1) \cdot 360360$.

Thus, $a_{15} \ge 1 + 14 \cdot 360360 = \boxed{5045041}$.

This bound is indeed attainable if we choose $a_i = 1+(i-1)\cdot 360360$: since the sum of any k such integers has the form $k + r \cdot 360360$ for some integer r, and 360360 is divisible by k for $1 \le k \le 15$, we conclude that any k of these integers has sum divisible by k and hence has an integral arithmetic mean.