Vermont Mathematics Talent Search, Solutions to Test 1, 2017-2018

Test and Solutions by Kiran MacCormick and Evan Dummit

November 21, 2017

1. Let S be a square and T be an equilateral triangle, and suppose S and T have the same perimeter. What is the ratio of their areas $\dfrac{\text{area}(S)}{\text{area}(T)}$?

   Answer: $3\sqrt{3}/4$.

   Solution: If the side length of the triangle is t, then its area is $\dfrac{t^2\sqrt{3}}{4}$ and its perimeter is $3t$. Therefore, the side length of the square is $\dfrac{3}{4}t$ and its area is $(\dfrac{3}{4}t)^2$. The desired ratio is then $\dfrac{(3t/4)^2}{t^2\sqrt{3}/4} = \dfrac{9/16}{\sqrt{3}/4} = \boxed{\dfrac{3\sqrt{3}}{4}}$.

2. The outside surface of a large cubical box is painted and then broken into unit cubes. If exactly 3174 of the unit cubes were painted on only one side, what is the side length of the large cube?

   Answer: 25.

   Solution: Consider the cubes that are painted on each face of the large cube: each of the unit cubes along the edge of each face will have more than one face painted (since they lie on two or three of the large cube's faces), while the cubes not bordering the edge will only have one face painted. Therefore, if the large cube has side length n, the cubes on each face that are only painted on one side will form an $(n-2)\times(n-2)$ square. Since there are six such faces, the number of small cubes painted on one side is $6(n-2)^2$. Solving $6(n-2)^2 = 3174$ produces $(n-2)^2 = 529$ so that $n-2 = \sqrt{529} = 23$ and thus $n = \boxed{25}$.

3. In how many ways can 4035 be written as the sum of two or more consecutive positive integers in increasing order? For example, 2017 + 2018 would count, but 2018 + 2017 would not.

   Answer: 7.

   Solution: Suppose that $a + (a+1) + \cdots + b = 4035$. By using the formula for the sum of an arithmetic progression, or writing $a+(a+1)+\cdots+b = [1+2+\cdots+b]-[1+2+\cdots+(a-1)] = \dfrac{b(b+1)}{2} - \dfrac{(a-1)a}{2}$ and factoring, we obtain the relation $\dfrac{1}{2}(a+b)(b-a+1) = 4035$. Equivalently, we see that $(a+b)(b-a+1) = 2\cdot 4035 = 2\cdot 3\cdot 5\cdot 269$. Now, if a, b are positive integers, we must have $a+b \geq b-a+1$, so making a table of the possibilities using the prime factorization yields the following:

$a+b$	$b-a+1$	a	b
8070	1	4035	4035
4035	2	2017	2018
2690	3	1344	1346
1614	5	805	809
1314	6	670	675
807	10	399	408
538	15	262	276
269	30	120	149

   Since the sum must contain more than one term, the first case $a = b = 4035$ is not acceptable, but the other $\boxed{7}$ cases work.

Remark: In general, the number of representations of n as the sum of two or more positive integers will depend on the prime factorization of n: if $n = 2^r k$ where k is odd, then the number of representations of n as a sum of at least two positive integers is equal to the number of divisors of k, minus 1.

4. If a and b are positive numbers and $a + b = ab = a^2 - b^2$, then what is a?

Answer: $a = \dfrac{3 + \sqrt{5}}{2}$.

Solution: If $a + b = a^2 - b^2 = (a - b)(a + b)$, then either $a + b = 0$ or $a - b = 1$. If $a + b = 0$, then the relation becomes $0 = -b^2 = 0$, so that $b = 0$, impossible. Therefore, $a - b = 1$, so $b = a - 1$ and then the relation becomes $2a - 1 = a(a - 1) = 2a - 1$, so we obtain the quadratic equation $a^2 - 3a + 1 = 0$. The solutions are $a = \dfrac{3 \pm \sqrt{5}}{2}$ yielding $b = \dfrac{1 \pm \sqrt{5}}{2}$, but since both a and b are positive we must have the plus sign. Thus, $a = \boxed{\dfrac{3 + \sqrt{5}}{2}}$.

5. How many ways are there to give 18 indistinguishable candy bars to Anisha, Boris, Chang, Diego, and Elsa in such a way that everyone gets at least one, and no one gets more than 6?

Answer: 780.

Solution 1: Suppose that Anisha receives A bars, Boris receives B, and so forth. We enumerate the solutions with $1 \le A \le B \le C \le D \le E \le 6$ and then count the possible ways of permuting the number of candy bars each person receives (which will only depend on how many people receive the same number):

$ABCDE$	#	$ABCDE$	#	$ABCDE$	#	$ABCDE$	#
11466	30	13356	60	22356	60	23445	60
11556	30	13446	60	22446	30	24444	5
12366	60	13455	60	22455	30	33336	5
12456	120	14445	20	23346	60	33345	20
12555	20	22266	10	23355	30	33444	10

Upon totaling the number of rearrangements, we see that the number of possible ways to distribute the bars is $\boxed{780}$.

Solution 2: As in the first solution, we wish to count the number of solutions to $A + B + C + D + E = 18$ where $1 \le A, B, C, D, E \le 6$. Observe that the result will be the coefficient of x^{18} in the expansion of $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5$, since, upon expanding the product, each term x^{18} arises as a product $x^A x^B x^C x^D x^E$ where $A + B + C + D + E = 18$ and $1 \le A, B, C, D, E \le 6$.

Expanding $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5 = x^5 \left(\dfrac{x^6 - 1}{x - 1}\right)^5$ by hand or via a computer yields the result $x^5 + 5x^6 + 15x^7 + \cdots + 780x^{17} + 780x^{18} + 735x^{19} + \cdots + 5x^{29} + x^{30}$, so the answer is $\boxed{780}$.

Solution 3: Suppose that Anisha receives $a + 1$ bars, Boris receives $b + 1$ bars, and so forth. We will first count all the possibilities in which $a + b + c + d + e = 13$, and then subtract the cases where one of a, b, c, d, e exceeds 5.

Consider the problem of arranging 13 stars and 4 separators in a row: there are $\binom{17}{4} = 2380$ such arrangements. If we let a be the number of stars before the first separator, b be the number of stars between the first and second separators, ... , and e be the number of stars after the last separator, then each arrangement of stars and separators yields one quintuple (a, b, c, d, e) with $a + b + c + d + e = 13$. Conversely, each such quintuple corresponds to a distinct arrangement of stars and separators, so the number of solutions to $a + b + c + d + e = 13$ with $a, b, c, d, e \ge 0$ is therefore 2380.

Now we must count the cases where one of a, b, c, d, e is at least 6. If two of a, b, c, d, e are at least 6, then either (i) one is 6, one is 7, and the others are 0, or (ii) two are 6, one is 1, and the others are 0. There are 20 possibilities in case (i) and 30 in case (ii).

Otherwise, exactly one of a, b, c, d, e is at least 6. By symmetry, we can assume that $a \ge 6$ and then multiply the total count by 5 when we are finished.

If $a = 6$, then $b + c + d + e = 7$ with $b, c, d, e \le 5$. By the same argument as given above, there are $\binom{10}{3} = 120$ solutions to $b + c + d + e = 7$, but we must subtract the 16 where one of b, c, d, e is 6 or 7 since those were already eliminated above. We obtain 104 solutions in this case.

If $a = 7$, then $b + c + d + e = 6$ with $b, c, d, e \leq 5$. By the same argument as given above, there are $\binom{9}{3} = 84$ solutions to $b + c + d + e = 6$, but we must subtract the 4 where one of b, c, d, e is 6. We obtain 80 solutions in this case.

For the remaining values of a, we do not need to subtract anything, so we get a total of $\binom{8}{3} + \binom{7}{3} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3} = 126$ solutions.

In total, we get $104 + 80 + 126 = 310$ solutions with $a \geq 6$ and $b, c, d, e \leq 5$, so the final answer is $2380 - 50 - 5 \cdot 310 = \boxed{780}$.

6. Suppose $p(x) = a_8x^8 + a_7x^7 + \cdots + a_2x^2 + a_1x + a_0$ is a polynomial with integer coefficients, where $a_8 \neq 0$. Prove that the maximum number of integers n that can satisfy $p(n)^2 = 25$ is 8.

Solution: Clearly, it is possible to have 8 integer solutions: for example, the polynomial $p(x) = (x - 1)(x - 2)(x - 3) \cdots (x - 8) + 5$ has $p(1) = p(2) = \cdots = p(8) = 5$.

We will use the following lemma to show that there cannot be more than 8 solutions:

Lemma: If $p(x)$ is a polynomial with integer coefficients and there is an integer solution to $p(n) = 5$, then there are at most 4 integer solutions to $p(n) = -5$.

Proof of Lemma: Suppose by way of contradiction that $p(k) = 5$ and that $p(n_1) = p(n_2) = \cdots = p(n_5) = -5$. Then there exists a polynomial $q(x)$ with integer coefficients such that $p(x) + 5 = q(x) \cdot (x - n_1)(x - n_2)(x - n_3)(x - n_4)(x - n_5)$. Setting $x = k$ then yields $10 = q(k) \cdot (k - n_1)(k - n_2)(k - n_3)(k - n_4)(k - n_5)$. Now notice that $k - n_1$, $k - n_2$, $k - n_3$, $k - n_4$, and $k - n_5$ are all distinct integers whose product divides 10, so they would have to be five of the numbers ± 1, ± 2, ± 5, and ± 10, but the product of any 5 of them has absolute value at least 20. This is a contradiction, so the lemma holds.

Now we count solutions to $p(n)^2 = 25$. If there are no integer solutions to $p(n) = 5$, there can be at most 8 integer solutions to $p(n)^2 = 25$, because the nonconstant degree-8 polynomial $p(x)$ can only take the value -5 at most 8 times. Similarly, if there are no integer solutions to $p(n) = -5$, there are also at most 8 solutions to $p(n)^2 = 25$. If both $p(n) = 5$ and $p(n) = -5$ have an integer solution, applying the lemma to $p(x)$ shows that there are at most 4 integer solutions to $p(n) = -5$. Also, applying the lemma to $-p(x)$ shows that there are at most 4 integer solutions to $p(n) = 5$; thus, again, there are at most 8 integer solutions to $p(n)^2 = 25$.