TALENT SEARCH 3 2012-2013 Solutions

1)) Given the array of letters shown, find the number of ways that the sequence C-A-R-R-O-L-L can be obtained if consecutive letters in the word are a single horizontal, vertical, or diagonal move away from each other.

O	L	L	L	L	L	O
L	R	R	R	R	R	L
L	R	A	A	A	R	L
L	R	A	C	A	R	L
L	R	A	A	A	R	L
L	R	R	R	R	R	L
O	L	L	L	L	L	O

An example is shown on the grid.

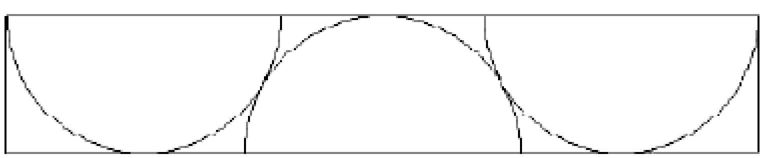
SOLUTION: **Answer 64**

There are 4 equivalent possible choices for O. Given an O, there are 4 possibilities for the OLL at the end. There are also 2 choices for the RRO in the middle, and the first R is always a knight's move (2 horizontal and 1 vertical, or the reverse) away from the starting C, meaning that there are 2 choices for the A. Hence there are a total of 4*4*2*2 = 64 choices.

2) Three non-overlapping semicircles of radius 1 are contained in a $1x\ d$ rectangle. Find the smallest possible value for d.

SOLUTION: **Answer $\boldsymbol{d = 2 + 2\sqrt{3}}$**

Clearly$d > 2$. Since the radii of the semicircles are equal to the height of the rectangle, the centers of the semicircles must lie on the long sides of the rectangle. If all 3 centers lie on the same side, $d = 6$ .If 2 centers lie on one side and 1 lies on the opposite side the distance between the centers of the circles is 2, and since the height of the rectangles is 1, the horizontal distance between the centers is $\sqrt{3}$. We obtain minimum$d = 2 + 2\sqrt{3}$.



3) How many different 3 x 3 arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, and the sums on the two major diagonals are 9 and 10.

SOLUTION: **Answer 3 unique arrays or 16 arrays given reflections and rotations**.

Let the entries in the array be $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then we may write the following system of equations.

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$$\begin{array}{ll} (1) & a_1 + a_2 + a_3 = 7 \\ (2) & a_4 + a_5 + a_6 = 7 \\ (3) & a_7 + a_8 + a_9 = 7 \\ (4) & a_1 + a_4 + a_7 = 7 \\ (5) & a_2 + a_5 + a_8 = 7 \\ (6) & a_3 + a_6 + a_9 = 7 \\ (7) & a_1 + a_5 + a_9 = 10 \\ (8) & a_3 + a_5 + a_7 = 9 \end{array}$$

By adding equations (1), (2), and (3) we get:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 21$$

Adding (2), (5), (7) and (8) we get:

$$a_1 + a_2 + a_3 + a_4 + 4a_5 + a_6 + a_7 + a_8 + a_9 = 33$$

Subtracting the two equations yields $3a_5 = 12$ and $a_5 = 4$. With no loss of generality, we can assume that $a_1 \geq a_9$ and $a_3 \geq a_7$ and from (7) $a_1 \geq 3$, and from (8) $a_3 \geq 3$. From (1) we can surmise there are only three possibilities for a_1 and a_3.

$$a_1 = a_3 = 3$$
$$a_1 = 3 \; and \; a_3 = 4$$
$$or \quad a_1 = 4 \; and \; a_3 = 3$$

Each of these leads to a unique array satisfying the requirements.

4) If the parabola $y = x^2 - c$ intersects the circle $x^2 + y^2 = c^2$in three distinct points forming an equilateral triangle, find all possible values of c.

SOLUTION: **Answer $c = 2$**

Setting $y = x^2 - c$ in $x^2 + y^2 = c^2$ yields $x^2 + (x^2 - c)^2 = c^2$, or $x^4 + (-2c + 1)x^2 = 0$. Thus the possible values for x are $x = 0$ (double root), $x = \sqrt{2c - 1}$, and $x = -\sqrt{2c - 1}$. Using $y = x^2 - c$ yields that the points of intersection are $A(0, -c)$, $B(\sqrt{2c - 1}, (c - 1)$,$C(-\sqrt{2c - 1}, c - 1)$.
In order for these points to form an equilateral triangle, we require all three distances between the points to be equal. Thus, setting AB = BC we obtain

$$\sqrt{(2c - 1)^2 + (2c - 1)^2} = \sqrt{\left(2\sqrt{2c - 1}\right)^2}$$

Equivalently after squaring, $2c - 1 + 4c^2 - 4c + 1 = 8c - 4$ or $4c^2 - 10c + 4 = 0$ which factors to $(2c - 1)(c - 2) = 0$

Solving gives $c = 2 \; or \; c = 1/2$ In the case where $c = 1/2$ there is only one one point of intersection between the circle and parabola. The solutions is therefore $c = 2$.

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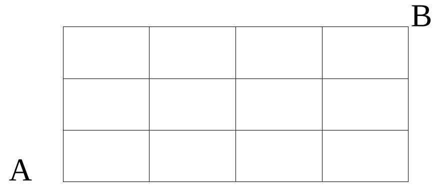
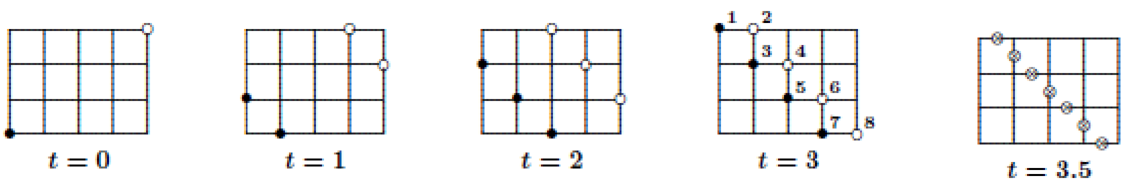
5) The figure below shows a street plan of twelve square blocks. A person P goes from point A to point B, and a second person Q goes from B to A. Both of them (P and Q) leave at the same time with the same speed, following shortest paths on the grid. At each corner they choose among the possible streets with equal probability. What is the probability that P meets Q?



SOLUTION: **Answer** $\frac{\mathbf{37}}{\mathbf{256}}$

The shortest path implies that P can only go up and right and Q can only go down or to the left so that each finishes the trip after travelling 7 blocks total (4 horizontal and 3 vertical)

Equate time with the number of blocks travelled. Illustrate the possible positions for both P (black circle and Q (open circle between times t = 0 and t = 3.



All 7 potential meetings take place at time t=3.5 at positions show in the diagram. The according probabilities are

$P_1 = P_7 = \frac{1}{8}$ and $P_3 = P_5 = \frac{3}{8}$ and $Q_2 = Q_8 = \frac{1}{8}$ and $Q_4 = Q_6 = \frac{3}{8}$

Let P_{ij} denote the probability that P moves from vertex I at time t=3 to vertex j at time t =4 and define Q_{ij} similarly. At vertices 3, 5 and 7 the probability of P going up or across are the same, whereas P can only go across from vertex 1. It follows that:

$P_{12} = \frac{1}{8}, P_{32} = P_{34} = P_{54} = P_{56} = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16} \, and \, P_{76} = P_{78} = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}.$

By a similar argument we obtain;

$Q_{87} = \frac{1}{8}, Q_{67} = Q_{65} = Q_{45} = Q_{43} = \frac{3}{16} \, and \, Q_{23} = Q_{21} = \frac{1}{16}.$

The probability that P meets Q is:

$$p = P_{12} \cdot Q_{21} + P_{32} \cdot Q_{23} + P_{34} \cdot Q_{43} + \cdots + P_{78} \cdot Q_{87}$$
$$= \frac{1}{8} \cdot \frac{1}{16} + \frac{3}{16} \cdot \frac{1}{16} + 3 \cdot \frac{3}{16} \cdot \frac{3}{16} + \frac{1}{16} \cdot \frac{3}{16} + \frac{1}{16} \cdot \frac{1}{8} = \frac{37}{256} \approx 14.45\%$$

6) For a positive number such as 3.14, we call 3 the *integer part*, and 0.14 the *fractional part*. Find a positive number such that the fractional part, the integer part, and the number itself are three consecutive terms

a. in an arithmetic sequence
b. in a geometric sequence.

Solution: **Answer a) 3/2 b)** $\mathbf{\frac{1+\sqrt{5}}{2}}$

Let *x* be a positive number. Let *n* be its integer part, and *y* its fractional part. Thus $x = n + y$, where *n* is an integer, and $0 \le y < 1$.

a. We want to find *x* so that *y, n,* and *x* are consecutive terms in an arithmetic sequence. First suppose that $0 < x < 1$. Then *n* = 0 and *x* = *y*. The numbers *y, n,* and *x* are not terms in an arithmetic sequence in this case.
Now assume $x \ge 1$. Then $n \ge 1$ and $y < n \le x$. In order for *y*, *n*, and *x* to be consecutive terms in an arithmetic sequence, we require

$$n - y = x - n = d,$$

for some real number d. Since $x = n + y$, we can eliminate *x* in the above equation to get $n - y = y = d$. Then $2y = n$. Since *n* is a positive integer and $0 \le y < 1$, this equation is satisfied only when $n = 1$ and $y = \frac{1}{2}$. Thus, the only possible value for *x* is $x = n + y = 1 + \frac{1}{2} = \frac{3}{2}$.

b. We want to find *x* so that *y, n*, and *x* are consecutive terms in a geometric sequence. As in part (a), we see that this does not happen if $0 < x < 1$. So we assume $x \ge 1$. Then $n \ge 1$ and $y < n \le x$.

In order for *y, n*, and *x* to be consecutive terms in a geometric sequence, we require

$$\frac{n}{y} = \frac{x}{n} = r,$$

for some $r \ne 0$. Since $x = n + y$, we can eliminate *x* in the above equation to get

$$\frac{n}{y} = 1 + \frac{y}{n} = r. \qquad (*)$$

Note that *y* cannot be 0, since this would make n/y undefined. So $0 < y < 1$. Therefore, $\frac{n}{y} > n$ and $1 + \left(\frac{y}{n}\right) < 1 + \left(\frac{1}{n}\right) \le 2$ since $n \ge 1$. Thus

$$n < \frac{n}{y} = 1 + \frac{y}{n} \le 2.$$

But we know n is a positive integer. The only possibility satisfying $n < 2$ is $n = 1$.

Setting $n = 1$ in (*), we get $\frac{1}{y} = 1 + y$, which can be rewritten as $y^2 + y - 1 = 0$. By the Quadratic Formula,

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}\ .$$

Since $y > 0$, we must have $y = \frac{-1+\sqrt{5}}{2}$. Thus, the only possible value for *x* is $x = n + y = 1 + \frac{-1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$.

7) Suppose we roll $N \geq 3$ standard 6-sided dice. What value(s) of N will maximize the probability of obtaining exactly three threes?

Solution**: Answer N=17 or N=18**

Let $f(n)$ be the probability of obtaining exactly three threes from N dice In order for there to be exactly three threes, there must be $N - 3$ dice not showing a 3. There are $\binom{N}{3}$ ways of picking the three 3's out of the N dice, so the probability of having exactly three 3's is $\binom{N}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^{N-3} = \frac{N(N-1)(N-2)}{6} \cdot \frac{5^{N-3}}{6^N}$. This expression is rather tricky to maximize directly. Let us instead consider the ratio $\frac{f(N)}{f(N-1)}$. If this ratio is greater than 1 then $f(N)$ is bigger than $f(N-1)$, and if this ratio is less than 1 the $f(N)$ is less than $f(N-1)$. We want to determine for which values of N this ratio is bigger than 1. We have $\frac{f(N)}{f(N-1)} = \frac{N(N-1)(N-2) \cdot 5^{N-3} / 6^N}{(N-1)(N-2)(N-3) \cdot 5^{N-4} / 6^{N-1}} = \frac{N \cdot 5/6}{N-3} = \frac{5}{6} \cdot \frac{N}{N-3}$, which is equal to 1 when $5N = 6N - 18$ or $N = 18$. From the expression we thus see that $\frac{f(N)}{f(N-1)} > 1$ for $N < 18$ and $\frac{f(N)}{f(N-1)} < 1$ for $N > 18$. Therefore we have $f(3) < f(4) < f(5) < \cdots < f(17) = f(18)$ and $f(18) > f(19) > f(20) > \cdots$ So there are two values of N which maximize the ratio, namely $N = 17$ and $N = 18$.

8) Find all x, $0 \leq x \leq \pi$, for which $\cos(x) \cdot \cos(2x) \cdot \cos(4x) = \frac{1}{8}$.

Solution: **Answer:** $\boldsymbol{x = \frac{\pi}{9}, \frac{2\pi}{7}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{4\pi}{7}, \frac{7\pi}{9}, \frac{6\pi}{7}}$

Multiply both sides by $sinx$;

$$\sin(x)\cos(x) \cdot \cos(2x) \cdot \cos(4x) = \frac{1}{8}\sin(x)$$

$$\frac{1}{2}\sin(2x)\cos(2x)\cdot\cos(4x)=\frac{1}{8}\sin(x)$$
$$\frac{1}{2}\sin(4x)\cos(4x)=\frac{1}{8}\sin(x)$$
$$\frac{1}{8}\sin(8x)=\frac{1}{8}\sin(x)$$

Therefore, we have *sin(x)=sin(8x)*.

By periodicity properties, we know that $\sin(\mathrm{a})=\sin(\mathrm{b})$ precisely when $a-b=2k\pi$ or $a+b=\pi+2k\pi$. Thus we either have, $7x=2k\pi\ or\ 9x=\pi+2k\pi$, so the possible solutions are $x=0,\frac{2\pi}{7},\frac{4\pi}{7},\frac{6\pi}{7}$, and $x=\frac{\pi}{9},\frac{3\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\pi$. Each value of *x* is a solution unless *sin(x)=0*, in which case the solutions may be extraneous because we started out by multiplying by *sin(x)*. In fact, for such *x*, we have $\cos(x)=\pm1$ and *cos(2x)*, *cos(4x)* are 1, so their product cannot be $\frac{1}{8}$. Therefore we have seven solutions: $x=\frac{\pi}{9},\frac{2\pi}{7},\frac{\pi}{3},\frac{5\pi}{9},\frac{4\pi}{7},\frac{7\pi}{9},\frac{6\pi}{7}$.