Talent Search Test 1 Solutions 2012

1) A 4 x 4 *antimagic* square is an arrangement of the numbers from 1 to 16 (inclusive) in a square, so that the totals of each of the four rows and four columns and two main diagonals are ten consecutive numbers in some order. The diagram below shows an incomplete anitmagic square. When it is completed what number will replace the star?

4	5	7	14
6	13	3	*
11	12	9	
10			

Solution: The numbers missing from the square are 1, 2, 8, 15, and 16. The completed rows, columns and diagonals add up as follows.
Row 1: 4+5+7+14=30, Column 1: 4+6+11+10 = 31, and the Diagonal: 10+12+3+14=39. The 10 sums must be the integers from 30 to 39. If we sum the partial rows, columns, diagonals we get: 6+13+3=22, 11+12+9=32, 5+13+12=30, 7+3+9=19 and 4+13+9=26. Using this process we can start to narrow down the possibilities and continue until we find the following solution.

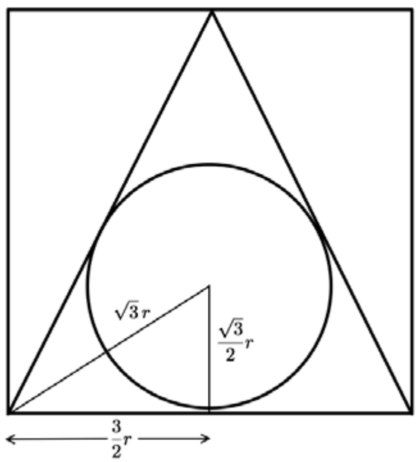
4	5	7	14
6	13	3	15
11	12	9	1
10	2	16	8

2) Suppose $p(x) = x^2 + cx + d$ satisfies $p(c) = p(d) = c$. If $cd \neq 0$, find p(2).

Solution: Since *p(c) =p (d)* we have $c^2 + c \cdot c + d = d^2 + c \cdot d + d, or\ 2c^2 - cd - d^2 = 0$.
Factoring gives $(2c + d)(c - d) = 0$. If $c = d, then\ p(c) = c$ requires $c^2 + c \cdot c + c = c\ or\ 2c^2 = 0$, which cannot happen. Thus, $d = -2c$. Then $p(c) = c$ requires
$2c^2 - 2c = c, so\ c = 0\ (not\ allowed)\ or\ c = \frac{3}{2}\ and\ d = -3.$ Then, $p(2) = 2^2 + \frac{3}{2} \cdot 2 - 3 = 4$.

3) A cube of side *2* is inscribed in a sphere. The sphere is inscribed in a cone with slant height equal to the diameter of its base. The cone is inscribed in a right circular cylinder. What is the surface area of the cylinder (including top and bottom)?

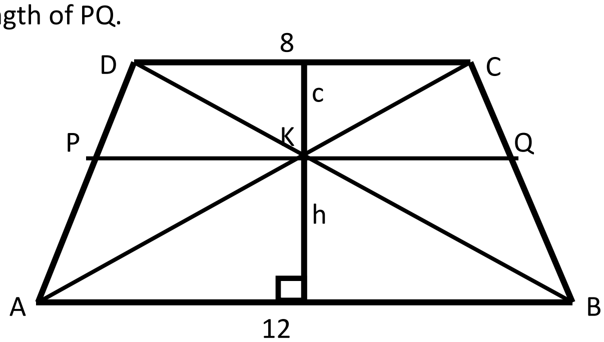
Solution: The sphere has diameter $2\sqrt{3}$ -- the length of the diagonal of the cube – hence it has radius $\sqrt{3}$. A simple diagram (below) shows that the cone must have base of diameter 6 – hence side length 6. (Because the lateral length and base of the cone are equal it is than a 30-60-90 triangle.



Thus the right circular cylinder has radius 3 and height $3\sqrt{3}$. In conclusion, the surface area of the cylinder is

$$2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height})$$
$$= 2\pi(3)^2 + 2\pi(3)\cdot 3\sqrt{3}$$
$$= 18\pi \cdot \left(1+\sqrt{3}\right)$$

4) ABCD is a trapezoid with AB || CD. Diagonals AC and BD intersect at k. The line through k parallel to AB intersects AD and BC at P and Q respectively. If AB = 12 and CD = 8, find the length of PQ.



Solution: Congruent Alternate Interior Angles: CDB and DBA and DKP, DCA and CAB, DCA and CKQ and CAB.

Therefore ΔCDK ~ ΔABK and $\frac{CD}{AB} = \frac{c}{h} = \frac{8}{12} = \frac{2}{3}$

Also, Δ PKD ~ ΔABD and $\frac{PK}{AB} = \frac{c}{c+h}$ *and using the above information,* $\frac{PK}{12} = \frac{c}{c+\frac{3c}{2}} = \frac{c}{\frac{5}{2}c} = \frac{2}{5}$

Therefore, $PK = \frac{24}{5}$. Also, Δ QKC ~ ΔBAC and by the same reasoning as above, $QK = \frac{24}{5}$. Finally,

PQ = PK + QK = 9.6

5) A die is tossed. If the die shows a 1 or a 2 then one coin is tossed. If the die shows a 3 then two coins are tossed. Otherwise, three coins are tossed. Given that the resulting coin tosses produced no heads, what is the probability that the die showed a 1 or a 2?

Solution: Let A be the event that the number 1 or 2 appears on the die, B the event that the number 3 appears and C the event that a number more than 3 appears. Then,

$P(A) = \frac{1}{3},\ P(B) = \frac{1}{6}, and\ P(C) = \frac{1}{2}$. Let NH be the event that no heads appear when the coins are tossed, then, $P(NH|A) = \frac{1}{2}, P(NH|B) = \frac{1}{4}, and\ P(NH|C) = \frac{1}{8}$. Therefore;

$$P(NH) = P(NH|A) \cdot P(A) + PNH|B) \cdot P(B) + P(NH|C) \cdot P(C)$$

$P(NH) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{8} \cdot \frac{1}{2} = \frac{13}{48}$ and we want to find P(A|NH) which is equal to

$$P(A|NH) = \frac{P(NH|A) \cdot P(A)}{P(NH)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{13}{48}} = \frac{8}{13}$$

6) Define the product (ꙍ) of two numbers as the sum of the product of the corresponding digits. So 246 ꙍ 738 = 2 x 7 + 4 x 3 + 6 x 8 =74. Find two numbers M and N so that M ꙍ N = 602 and M + N is a minimum.

Solution: M and N will have the same number of digits, n, because if a digit was multiplied by zero it would not matter to the product.

M ꙍ N = $M_n N_n + M_{n-1} N_{n-1} + \cdots + M_1 N_1 \ \leq\ 9 \cdot 9 + 9 \cdot 9 + \cdots + 9 \cdot 9$

Therefore, $602 \leq 81n\ or\ n \geq 8$ since n must be an integer. 8 times 81 is greater than 2002 so we know we have to have n = 7 and $602 - 7 \cdot 81 = 35$. To be able to get 35 and keep it as a minimum, we can use $7 \cdot 5 = 35$. Therefore the answer which keeps M+N a minimum is:

79,999,999 and 59,999,999

The sum of the digits is simply, 7 + 5 + (14)(9) = 138

7) Find all positive integers *n* that are within 201.3 of exactly 15 perfect squares. *Note*: The perfect squares are {0, 1, 4, 9 …}. The within is not strict, that is *n* is within *r* of *m* if the difference between *n* and *m* is less than **or equal** to r. (e.g. 300 is within 100 of 200).

Solution: We seek all positive integers *n* such that the closed interval $I = [n - 201.3, n + 201.3]$ contains exactly 15 perfect squares. Of course, such squares would have to be consecutive, and so, for some positive integer $x \geq 7$, they would be the 15 squares,

$$(x-7)^2, (x-6)^2, \dots, x^2, \dots, (x+7)^2$$

In order for the interval *I*, whose length is 402.6, to contain these squares, it must be that

$$(x+7)^2-(x-7)^2 \le 402.6,$$

$$28x \le 402.6 \Longrightarrow x \le 14,$$

since *x* is an integer. Therefore the squares in question must be 15 consecutive values from the squares up to $(14+7)^2 = 21^2$: $\{0, 1, 4, \ldots, 400, 441\}$.

Now for the interval *I* to contain precisely the 15 squares

$$(k+1)^2, (k+2)^2, \ldots, (k+15)^2,$$

the interval cannot extend as far as the next square on either side of them, that is, it can't reach as far as k^2 on the left nor as far as $(k+16)^2$ on the right. Hence *I* is contained in the open interval $(k^2, (k+16)^2)$, and we have

$$\begin{aligned} [(k+16)^2]-k^2 &> 402.6, \\ 32k+256 &> 402.6, \\ k &\ge 5. \end{aligned}$$

Thus the smallest of the 15 squares must be at least $(k+1)^2 = 6^2 = 36$, which restricts the string of squares to the two sets (i) {36, 49, 64, …, 400} or (ii) {49, 64, …, 441}

For set (i) to lie in the interval $I = [n-201.3, n+201.3]$, we must have

$$n-201.3 > 5^2 = 25 \quad \text{and} \quad n+201.3 < 21^2 = 441,$$

$$n \ge 227 \quad \text{and} \quad n \le 239,$$

That is, $n = 227, 228, \ldots, 239$.

All but the last two values are acceptable since they yield the intervals [25.7, 428.3], [26.7, 429.3], [27.7, 430.3], …, [35.7, 438.3], each of which contains set (i), but *n* = 238 yields the interval [36.7, 439.3], and *n* = 239 yields [37.7, 440.3], which do not.

Similarly, for set (ii), {49, 64, …, 441}, we must have

$$n-201.3 > 36 \quad \text{and} \quad n+201.3 < 484,$$

$$n \ge 238 \quad \text{and} \quad n \le 282,$$

suggesting $n = 238, 239, 240, \ldots, 281$. We've already established 238 and 239 do not work. However, to include the square 49, *n* cannot exceed 49+201.3 = 250.3. Therefore this case actually yields only the values $n = 240, \ldots, 250$, and since the corresponding intervals

$$[38.7, 441.3], [39.7, 442.3], \ldots, [48.7, 451.3]$$

all contain the set (ii), we conclude that *n* may be any integer from 228 to 250, inclusive, except for 238 and 239.

8) A real number is called "integrally radical" if it is the sum of some number of terms of the form $\sqrt{k}$ and $-\sqrt{k}$, where *k* is a positive integer. For example, $3 = \sqrt{9}, \sqrt{2} - \sqrt{3}$, and $\frac{6}{\sqrt{2}} = \sqrt{2} + \sqrt{2} + \sqrt{2}$ are integrally radical, but $\frac{1}{3}, \frac{\sqrt{3}}{2}, and\ \frac{5}{\sqrt{2}}$ are not. Find the smallest positive integer *n* such that $\frac{n}{\sqrt{5}+\sqrt{6}+\sqrt{7}}$ is integrally radical.

Solution: First rationalize the denominator,

$$\frac{1}{\sqrt{5}+\sqrt{6}+\sqrt{7}} = \frac{1}{\sqrt{5}+\sqrt{6}+\sqrt{7}} \cdot \frac{\sqrt{5}+\sqrt{6}-\sqrt{7}}{\sqrt{5}+\sqrt{6}-\sqrt{7}} = \frac{\sqrt{5}+\sqrt{6}-\sqrt{7}}{\left(\sqrt{5}+\sqrt{6}\right)^2 - \left(\sqrt{7}\right)^2} = \frac{\sqrt{5}+\sqrt{6}-\sqrt{7}}{4+2\sqrt{30}}$$

Repeating the process: $\frac{1}{4+2\sqrt{30}} \cdot \frac{4-2\sqrt{30}}{4-2\sqrt{30}} = \frac{4-2\sqrt{30}}{4^2-4\cdot 30} = \frac{4-2\sqrt{30}}{-104}$. Therefore, $\frac{1}{\sqrt{5}+\sqrt{6}+\sqrt{7}} = \frac{\left(\sqrt{5}+\sqrt{6}-\sqrt{7}\right)\left(4-2\sqrt{30}\right)}{-104}$.

So, $\frac{-104}{\sqrt{5}+\sqrt{6}+\sqrt{7}} = \left(\sqrt{5}+\sqrt{6}-\sqrt{7}\right)\left(4-2\sqrt{30}\right) = -8\sqrt{5} - 6\sqrt{6} - 4\sqrt{7} + 2\sqrt{210}$. Dividing by -2 gives,

$\frac{52}{\sqrt{5}+\sqrt{6}+\sqrt{7}} = 4\sqrt{5} + 3\sqrt{6} + 2\sqrt{7} - \sqrt{210}$. From this expression we see that the minimal value of *n* is 52.