

Vermont Talent Search
 School Year 2010-2011
 Test 4 Solutions

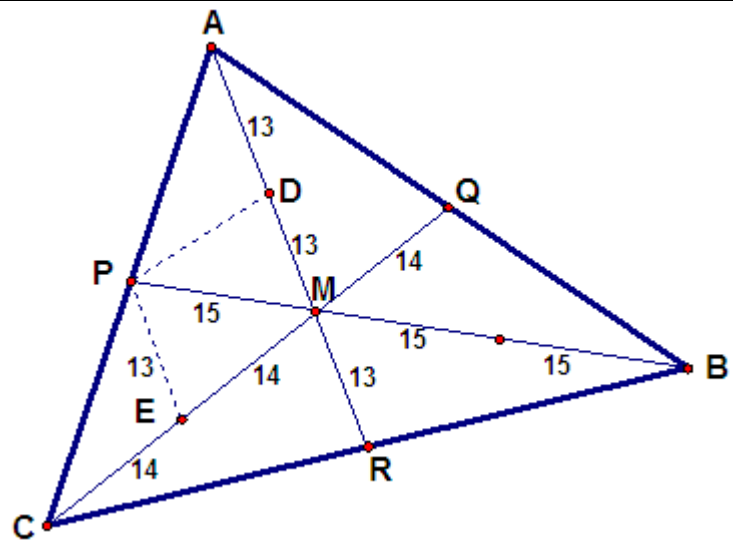
April 12, 2011

Problem 1.

Find the area of a triangle whose medians have lengths of 39, 42 and 45.

Solution:

Let M be the center of gravity or centroid of the triangle. Medians intersect at a point $\frac{2}{3}$ the distance from the vertices. Let D bisect AM and E bisect CM . In $\triangle ACM$, PD and PE are midlines, i.e. join midpoints of sides, thus $PD = 14$ and $PE = 13$. $\triangle DMP$ and $\triangle EMP$ are 13,14,15 triangles and their areas are 84. (you can verify using Hero's formula) Since D is a midpoint, the areas of $\triangle ADP$ and $\triangle DMP$ are equal. Thus each of the 4 triangles in $\triangle ACM$ has area 84. Similar results occur in $\triangle BCM$ and $\triangle ABM$. Therefore,



area of $\triangle ABC = 3 \cdot 4 \cdot 84 = 1008$.

Problem 2.

In quadrilateral $ABCD$, $\cot A = 4$, $\cot B = \frac{3}{2}$, $\cot C = 5$. Find all possible values for $\cot D$.

Solution:

Let $a = \tan A$, $b = \tan B$, $c = \tan C$ and $d = \tan D$

Then $a = \frac{1}{4}$, $b = \frac{2}{3}$ and so on.

We have $A + B + C + D = 2\pi$ and hence $\tan(A + B + C + D) = 0$.

Writing $A + B + C + D$ as $(A + B) + (C + D)$ and applying $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

yields: $\tan(A + B) + \tan(C + D) = 0$. Now applying the sum relation to both sums gives:

$$\frac{a+b}{1-ab} + \frac{c+d}{1-cd} = 0. \text{ Now cross multiply to get: } a+b-acd-bcd+c+d-abc-abd = 0$$

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Solving for d gives $d = \frac{a+b+c-abc}{ab+ac+bc-1}$. Inserting values for a, b, c yields $d = -\frac{65}{39} = -\frac{5}{3}$
 and hence $\cot D = -\frac{3}{5}$.

Problem 3.

In $\triangle ABC$, $AB = 4, BC = 5$ and $AC = 6$. Equilateral triangles ABD and CBF are drawn exterior to triangle ABC . CD and AF are drawn. Find the sum of $X + Y$ where $X = \angle ACD$ and $Y = \angle CAF$

Solution:

<p> $\angle DBC = \angle ABF ; (\angle B + 60^\circ)$ $BD = BA = 4$ and $BF = BC = 5$ $\triangle BCD \cong \triangle ABF$ by SAS and therefore $\angle 1 = \angle 2$ and since vertical angles are equal, $\triangle BFQ \cong \triangle PCQ$. Thus $\angle FBQ = \angle CPQ = 60^\circ$ But $\angle CPQ$ is an exterior angle for $\triangle ACP$ and $X + Y = 60^\circ$ </p>	
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Problem 4.

Given $f(n) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$.

If $f(n+1) - f(n-1) = kf(n)$ where k is an integer, find k .

Solution:

From the given equation we have

$$f(1) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)$$

$$f(-1) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{-1} + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{-1}$$

$$kf(0) = k\left(\frac{5+3\sqrt{5}}{10}\right) + k\left(\frac{5-3\sqrt{5}}{10}\right)$$

The equation is true for all n , so let $n = 0$, then $f(1) - f(-1) - kf(0) = 0$.

Therefore, factoring out terms we have..

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$$\left(\frac{5+3\sqrt{5}}{10}\right)\left[\frac{1+\sqrt{5}}{2}-\left(\frac{1+\sqrt{5}}{2}\right)^{-1}-k\right]+\left(\frac{5-3\sqrt{5}}{10}\right)\left[\frac{1-\sqrt{5}}{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{-1}-k\right]=0$$

For the equation to be zero, the quantities in the brackets must be zero. Therefore

$$\frac{1+\sqrt{5}}{2}-\frac{2}{1+\sqrt{5}}-k=0 \quad \text{or} \quad (6+2\sqrt{5}-4)-2k(1+\sqrt{5})=0$$

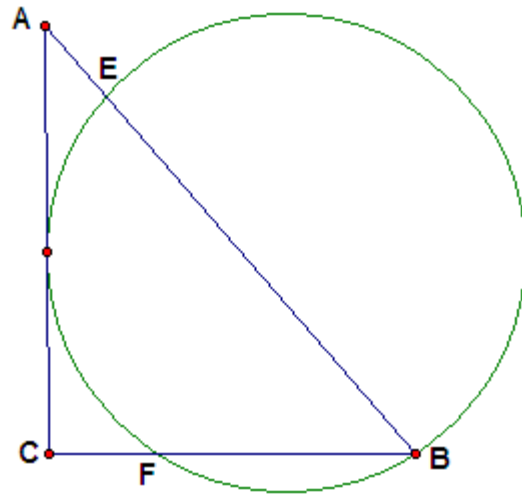
Thus $(2+2\sqrt{5})(1-k)=0$ and $k=1$

Problem 5.

A circle is tangent to leg AC of right $\triangle ABC$ and intersects the hypotenuse AB at E and leg BC at F . Point B is on the circle's circumference.

AC and BC have integral lengths and $AC > BC$.

If $AE = 4$ and $BE = 21$ find the radius of the circle.



Solution:

Draw radii OD , OF , and OB . $OD \perp AC$.

Draw DF . Let $\angle DOF = \angle BFO = \theta$

Note that $\triangle AED \sim \triangle ADB$. Thus

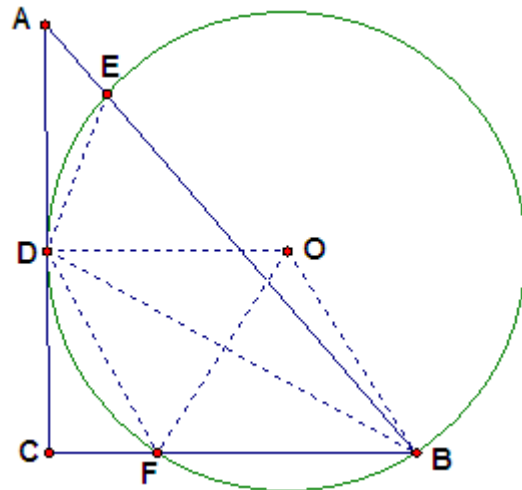
$$(AD)^2 = AE \cdot AB = 4 \cdot 25 = 100 \quad \text{and} \quad AD = 10$$

The only right \triangle 's with hypotenuse of 25 are $(7,24,25)$ and $(15,20,25)$ and $(7,24,25)$ is not possible. Thus $AC=20$, $CD=10$ & $BC=15$.

Note also that $\triangle DCF \sim \triangle DCB$; thus

$$CF = \frac{20}{3}. \quad \text{Hence}$$

$$DF = \sqrt{100 + \frac{400}{9}} = \frac{10\sqrt{13}}{3}$$



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Now use Law of Cosines in triangle DOF : $(DF^2) = r^2 + r^2 - 2r^2 \cos \theta$

and $\cos \theta = \frac{18r^2 - 1300}{18r^2}$. But in triangle BOF , which is also isosceles, $\cos \theta = \frac{25}{6r}$.

Equating these relationships and simplifying gives $18r^2 - 75r - 1300 = 0$

$$(3r + 20)(6r - 65) = 0$$

$$\text{and } r = \frac{65}{6}$$

Problem 6.

Two numbers from the set $S = \{1, 2, 3, \dots, 106\}$ are selected at random and multiplied. What is the probability that the product is a multiple of 5.

Solution:

Any number that is a multiple of 5 will produce a product that is a multiple of 5. There are 21 numbers in S that are multiples of 5 and 85 that are not multiples of 5.

Probability first number selected is multiple of 5 is $\frac{21}{106}$.

If there is no replacement, the second number is immaterial.

Probability first number not multiple of 5, but second one is equals $\frac{85}{106} \cdot \frac{21}{105}$

In this case, the probability product is a multiple of 5 equals: $\frac{21}{106} + \frac{85}{106} \cdot \frac{21}{105} = \frac{19}{53}$

With the first number replaced, the probability of either number being a multiple of 5 is $\frac{21}{106}$.

Equivalently, the probability of neither number being a multiple of 5 is $\left(\frac{85}{106}\right)^2$. Therefore

the probability the product is a multiple of 5 is $\left(1 - \left(\frac{85}{106}\right)^2\right) = \frac{4011}{11236}$

Note: Due to the inexact problem statement, either answer or both receive full credit.

Problem 7

Let a, b, c , and d be positive real numbers such that $\log_a b = c$, $\log_b c = 2d$, $\log_c d = 3a$, and $\log_d a = 4b$. Find the numerical value of the product $abcd$.

Solution:

From the given information we can say as follows:

$$a^c = b, b^{2d} = c, c^{3a} = d, \text{ and } d^{4b} = a$$

Thus the product

$$abcd = d^{4b} \cdot a^c \cdot b^{2d} \cdot c^{3a} = d^{4b} \cdot b \cdot b^{2d} \cdot d = b^{2d+1} \cdot d^{4b+1} = a^{c^{2d+1}} \cdot c^{(3a)^{4b+1}}$$

$$abcd = a^{(2cd+c)} \cdot c^{(12ab+3a)} = a^{2cd} \cdot a^c \cdot c^{12ab} \cdot c^{3a} = a^{2cd} \cdot b \cdot c^{12ab} \cdot d$$

$$ac = a^{2cd} \cdot c^{12ab} \text{ so } 2cd = 1 \text{ or } cd = \frac{1}{2} \text{ and } 12ab = 1 \text{ so } ab = \frac{1}{12} \quad abcd = \frac{1}{24}$$

Problem 8.

Find the sum $\sum_S \frac{1}{st}$ where S is the collection of all ordered pairs (s, t) of relatively prime positive integers such that $0 < s < t \leq 12$ and $s + t > 12$.

Solution:

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The following solution will show by mathematical induction that $\sum_s \frac{1}{st} = \frac{1}{2}$ for any n .

Basic Step: Show that $\sum_s \frac{1}{st}$ is true for the base value $n = 2$. Clearly the only term in the sum is $(s, t) = (1, 2)$ in which case the sum is $\frac{1}{2}$.

Inductive Step: Assume that $\sum_s \frac{1}{st} = \frac{1}{2}$ for $\nu = n$ and prove that $\sum_s \frac{1}{st} = \frac{1}{2}$ for $\nu = n + 1$.

Specifically, the inductive step goes from adding all terms with $0 < s < t \leq n$ and $s + t > n$ to instead adding all terms with $0 < s < t \leq n + 1$ and $s + t > n + 1$. In doing so all terms with $s + t = n + 1$ are removed and all terms with $t = n + 1$ are added.

We claim the net effect of doing this is zero. To see this we need to show that

For $\nu < n + 1 - \nu$

$$\sum_{\substack{\gcd(\nu, n+1-\nu)=1 \\ \nu < n+1-\nu}} \frac{1}{\nu(n+1-\nu)} = \sum_{\gcd(s, n+1)} \frac{1}{(n+1)s} \quad \text{EQ. 1}$$

where the LHS of EQ 1 is the sum of the removed terms and the RHS is the sum of the additional terms. Also, observe that ν and $n + 1 - \nu$ are relatively prime if and only if ν and $n + 1$ are relatively prime. So we are summing over the same range.

Writing the summand on the LHS as $\frac{1}{\nu(n+1-\nu)} = \frac{1}{n+1} \left[\frac{1}{\nu} + \frac{1}{n+1-\nu} \right]$ and splitting into two sums gives the LHS as

$$\sum_{\substack{\gcd(\nu, n+1-\nu)=1 \\ \nu < n+1-\nu}} \frac{1}{\nu(n+1)} + \sum_{\substack{\gcd(\nu, n+1-\nu)=1 \\ \nu < n+1-\nu}} \frac{1}{(n+1)(n+1-\nu)}$$

Now interchange ν and $n + 1 - \nu$ in the second sum which gives just

$$\sum_{\substack{\gcd(\nu, n+1)=1 \\ \nu > n+1-\nu}} \frac{1}{(n+1)\nu} \quad \text{which is the "missing half" of the sum on the other side.}$$

So the net effect of going from n to $n + 1$ is zero as claimed.

The following spreadsheet shows this in the case of $0 < s < t \leq 12$ and $s + t > 12$.

N=12

	S	T	S+T	(ST) ⁻¹	
	1	12	13	0.083333	12
1	2	11	13	0.045455	22
2	3	11	14	0.030303	33
3	3	10	13	0.033333	30
4	4	11	15	0.022727	44
5	4	9	13	0.027778	36
6	5	12	17	0.016667	60
7	5	11	16	0.018182	55
8	5	9	14	0.022222	45
9	5	8	13	0.025	40
10	6	11	17	0.015152	66
11	6	7	13	0.02381	42
12	7	12	19	0.011905	84
13	7	11	18	0.012987	77
14	7	10	17	0.014286	70
15	7	9	16	0.015873	63
16	7	8	15	0.017857	56
17	8	11	19	0.011364	88
18	8	9	17	0.013889	72
19	9	11	20	0.010101	99
20	9	10	19	0.011111	90
21	10	11	21	0.009091	110
22	11	12	23	0.007576	132
23					
24			sum=	0.5	
25					
26					
27					

N=13

	S	T	S+T	(ST) ⁻¹	
	1	13	14	0.076923	
	2	13	15	0.038462	26
	3	13	16	0.025641	39
	3	11	14	0.030303	33
	4	13	17	0.019231	52
	4	11	15	0.022727	44
	5	13	18	0.015385	65
	5	12	17	0.016667	60
	5	11	16	0.018182	55
	5	9	14	0.022222	45
	6	13	19	0.012821	78
	6	11	17	0.015152	66
	7	13	20	0.010989	91
	7	12	19	0.011905	84
	7	11	18	0.012987	77
	7	10	17	0.014286	70
	7	9	16	0.015873	63
	7	8	15	0.017857	56
	8	13	21	0.009615	104
	8	11	19	0.011364	88
	8	9	17	0.013889	72
	9	13	22	0.008547	117
	9	11	20	0.010101	99
	9	10	19	0.011111	90
	10	13	23	0.007692	130
	10	11	21	0.009091	110
	11	13	24	0.006993	143
	11	12	23	0.007576	132
	12	13	25	0.00641	156

sum= 0.5

removed terms= 0.238709

added terms= 0.238709

Note: Correct solutions supported with sound mathematical reasoning received full credit. It was not necessary to provide the solution by mathematical induction as shown above.

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