Vermont State Mathematics Coalition Talent Search February 8, 2010

Test 3 Solutions

Problem 1.
Find all real values of x satisfying $x^{2\log_2 x} = 8$.
Solution:
Remembering that $y = \log_a x$ is equivalent to $a^y = x$; rewrite the given equation (by taking base 2 logarithms of both sides) as:

$(2\log_2 x)(\log_2 x) = 3$; thus $(\log_2 x)^2 = 3/2$ and $\log_2 x = \pm\sqrt{\frac{3}{2}}$. Finally $x = 2^{\pm\sqrt{\frac{3}{2}}}$

Problem 2.
A natural number x, all of whose digits are nonzero, satisfies the equation $x \cdot \overline{x} = 1000 + p(x)$, where $\overline{x}$ is the number *x* with its digits reversed, and *p(x)* is the product of the digits of *x*. Find all possible numbers *x*.
Solution:
First realize *x* must be a two digit number; therefore we can write
$(10t+u)(10u+t) = 1000 + tu$
Thus, $100tu + 10t^2 + 10u^2 + tu = 1000 + tu$ or $10tu + t^2 + u^2 = 100$
By inspection, we conclude $t, u \le 6$
If $t = 6$, u must be 1, but this is not a solution to the required equation.
If $t = 5$ u must be 1, but this is not a solution to the required equation.
If $t = 4$, $u = 2$ is a solution as is $t = 2$, $u = 4$.
If $t = 3$, $u = 1$ or 2 provides no solution.
Thus $x = 42$ or 24.

Problem 3.
A line from ***a*** to ***b*** has midpoint at ***c***. A point is chosen at random on the line and marked ***x***. Find the probability that the line segments *ax, bx*, and *ac* can be joined to form a triangle.
Solution:
Without loss of generality, let $a = 0$ and $b = 1$ and ***x*** a random variable $0 \le x \le 1$.
For $x > 0.5$, ax will be the longest side of a potential triangle. Using the triangle inequality (no side can be as long as the sum of the other two sides) we know $ax < bx + ac$ or $x < (1-x) + 0.5$ which gives $x < 0.75$
For $x < 0.5$, *bx* will be the longest side so $bx < ax + ac$ or $1 - x < x + 0.5$ so $x > 0.25$.
So values of ***x*** that permit construction of a triangle are $0.25 < x < 0.75$ which is half of the ***x*** values so probability of triangle construction is $\frac{1}{2}$.

Problem 4.

Find the smallest natural number $\boldsymbol{n}$ for which $\sum_{i=1}^{n} i = \sum_{i=1}^{k} (n+i)$ for $k > 1$.

Solution:

Find the sums by multiplying the number of terms by the average of the numbers.

For $\sum_{i=1}^{n} i = 1+2+3+...+n = \dfrac{n(n+1)}{2} = \dfrac{n^2+n}{2}$.

For $\sum_{i=1}^{k} (n+i) = (n+1)+(n+2)+...+(n+k) = \dfrac{(n+1+n+k)k}{2}$

Thus $\dfrac{n^2+n}{2} = \dfrac{2nk+k+k^2}{2}$ and $n^2 + (1-2k)n - k - k^2 = 0$

Solving for $\boldsymbol{n}$; $n = \dfrac{-(1-2k) \pm \sqrt{(1-2k)^2 + 4(k+k^2)}}{2} = \dfrac{2k-1 \pm \sqrt{8k^2+1}}{2}$

But $(8k^2+1)$ must be a perfect square for $k > 1$. Find smallest k to be $k = 6$

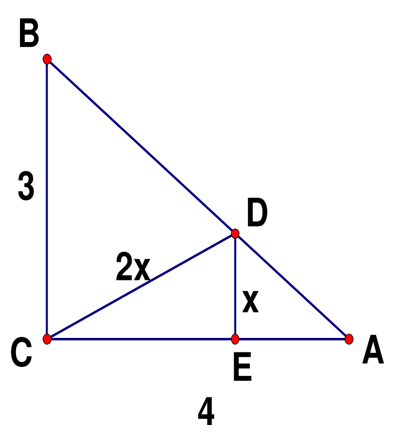
And $n = \dfrac{11 \pm 17}{2}$ or $n = 14$. Verify: $\sum_{1}^{14} i = 105$ and $\sum_{1}^{6} 14 + i = 15 + 16 + ... + 20 = 105$

Problem 5.

Triangle *ABC* is a right triangle with legs *BC* = *3* and *AC* = *4*. The length of the longer angle trisector from *C* to the hypotenuse is $\dfrac{a\sqrt{3}+b}{c}$. Evaluate $11a - 10b + 3c$.

Solution:

Draw *CD* with $\angle ACD = 30$. Draw $DE \perp AC$. Hence ΔCDE is a 30-60-90 triangle.



Let $DE = x$, $CD = 2x$ $CE = x\sqrt{3}$; then $AE = 4 - x\sqrt{3}$.

$\Delta ADE \sim \Delta ABC$ and $\dfrac{x}{4 - x\sqrt{3}} = \dfrac{3}{4}$;hence

$x = \dfrac{12}{3\sqrt{3}+4}$ and $2x = \dfrac{24}{3\sqrt{3}+4}$

$$\frac{24}{3\sqrt{3}+4}\left(\frac{3\sqrt{3}-4}{3\sqrt{3}-4}\right) = \frac{72\sqrt{3}-96}{11}$$

Therefore $a = 72$, $b = -96$, $c = 11$ and $11a - 10b + 3c = 1785$

Problem 6.

Let the digital sum of a number be defined as the base 10 sum of its digits. Thus 400_{10} has a digital sum of 4, but $400_{10} = 294_{12}$ has a digital sum of 15 and $400_{10} = 110010000_2$ has a digital sum of 3. Certain numbers, when written is base 4, have a digital sum of 17. Let the digital sum of such a number, when the number is written is base 2, be K. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are the minimum and maximum possible values of K respectively, compute the ordered pair (***x***,***y***).

Solution:
When a number changes from base 4 to base 2, all 0's and 1's remain 0's and 1's (although the 1's change position); 2's change to 1's (in different positions) and 3's change to 11's. Example: $123_4 = 11011_2$.. Thus each 2 or 3 "loses" one in the new digital sum. (No new 1's will ever overlap to form a 2, which then becomes a 1 in another position). Therefore the maximum value of K occurs when the base 4 representation consists of no digits greater than 1; then y = 17. Similarly, the minimum occurs when as many 2's and 3's as possible occur in the base 4 representation. Using eight 2's and one 1 (or seven 2's and three 1's) produces a digital sum *loss* of 8, so $x = 17 - 8 = 9$. Therefore ordered pair is $(9,17)$.

Problem 7.
At a high school in Germany, 80% of all students in the school are male, but only 5% of the male students and 10% of the female students are on the math team. If a member of the math team is selected at random, with each member of the team having an equal chance of being selected, compute the probability that the selected member is male.

Solution:
It is easiest to assume there are 100 students is the school. Then 80 will be male and 20 will be female. That means the math team consists of 4 males (5%) and 2 females (10%); thus the required probability is 4/6 = 2/3.

Problem 8.
What is the largest integer multiple of 8, no two of whose digits are the same?

Solution:
A number is divisible by 8 if and only if its last three digits are divisible by 8. The largest integer having no two digits the same is of course 9876543210. Rearranging the last three digits gives the largest such integer divisible by 8, namely 9876543120.

Special Note:

1. **The fourth test will be available on March 10, 2010 at www.vtmathcoalition.org**

The Math Coalition is grateful for problem contributors for this test including Middlebury College professors Michael Olinick, Bill Peterson, and Peter Schumer. Also contributing is Tony Trono, retired Burlington High School math teacher and Evan Dummit a graduate mathematics student at the California Institute of Technology.